

# Nonrepetitive Colourings of Planar Graphs with $O(\log n)$ Colours

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## Abstract

A vertex colouring of a graph is *nonrepetitive* if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. The *nonrepetitive chromatic number* of a graph  $G$  is the minimum integer  $k$  such that  $G$  has a nonrepetitive  $k$ -colouring. Whether planar graphs have bounded nonrepetitive chromatic number is one of the most important open problems in the field. Despite this, the best known upper bound is  $O(\sqrt{n})$  for  $n$ -vertex planar graphs. We prove a  $O(\log n)$  upper bound.

## 1 Introduction

A vertex colouring of a graph is *nonrepetitive* if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. More precisely, a  $k$ -colouring of a graph  $G$  is a function  $\psi$  that assigns one of  $k$  colours to each vertex of  $G$ . A path  $(v_1, v_2, \dots, v_{2t})$  of even order in  $G$  is *repetitively* coloured by  $\psi$  if  $\psi(v_i) = \psi(v_{t+i})$  for all  $i \in [1, t] := \{1, 2, \dots, t\}$ . A colouring  $\psi$  of  $G$  is *nonrepetitive* if no path of  $G$  is repetitively coloured by  $\psi$ . Observe that a nonrepetitive colouring is *proper*, in the sense that adjacent vertices are coloured differently. The *nonrepetitive chromatic number*  $\pi(G)$  is the minimum integer  $k$  such that  $G$  admits a nonrepetitive  $k$ -colouring.

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The seminal result in this field is by Thue [38], who in 1906 proved that every path is nonrepetitively 3-colourable. Nonrepetitive colourings have recently been widely studied [2–9, 11–13, 19, 21, 23, 25–32, 34–37]; see the surveys [10, 22–24]. A number of graph classes are known to have bounded nonrepetitive chromatic number. In particular, trees are nonrepetitively 4-colourable [8, 30], outerplanar graphs are nonrepetitively 12-colourable [5, 30], and more generally, every graph with treewidth  $k$  is nonrepetitively  $4^k$ -colourable [30]. Graphs with maximum degree  $\Delta$  are nonrepetitively  $O(\Delta^2)$ -colourable [3, 22, 23, 27].

Perhaps the most important open problem in the field of nonrepetitive colourings is whether planar graphs have bounded nonrepetitive chromatic number. This question, first asked by Alon et al. [3], has since been mentioned by numerous authors [2, 5, 13, 21–24, 26, 27, 30, 32, 34]. It is widely known that  $\pi(G) \in O(\sqrt{n})$  for  $n$ -vertex planar graphs<sup>1</sup>, and this is the best known upper bound. The best known lower bound is 11, due to Pascal Ochem; see Appendix A. Here we prove a logarithmic upper bound.

**Theorem 1.** *For every planar graph  $G$  with  $n$  vertices,*

$$\pi(G) \leq 8(1 + \log_{3/2} n) .$$

As a secondary contribution, we solve the above open problem when restricted to paths of bounded length.

**Theorem 2.** *There is a constant  $c$  such that, for every integer  $k \geq 1$ , every planar graph  $G$  is  $c^{k^2}$ -colourable such that  $G$  contains no repetitively coloured path of order at most  $2k$ .*

Note that the case  $k = 2$  of Theorem 2 corresponds to so-called *star colourings*; that is, proper colourings with no 2-coloured  $P_4$ ; see [1, 18, 33, 39]. Albertson et al. [1] proved that every planar graph is star colourable with 20 colours.

## 2 Proof of Theorem 1

A *layering* of a graph  $G$  is a partition  $V_0, V_1, \dots, V_p$  of  $V(G)$  such that for every edge  $vw \in E(G)$ , if  $v \in V_i$  and  $w \in V_j$  then  $|i - j| \leq 1$ . Each set  $V_i$  is called a *layer*. The following lemma by Kündgen and Pelsmajer [30] will be useful.

**Lemma 3** ([30]). *For every layering of a graph  $G$ , there is a (not necessarily proper) 4-colouring of  $G$  such that for every repetitively coloured path  $(v_1, v_2, \dots, v_{2t})$ , the subpaths  $(v_1, v_2, \dots, v_t)$  and  $(v_{t+1}, v_{t+2}, \dots, v_{2t})$  have the same layer pattern.*

A *separation* of a graph  $G$  is a pair  $(G_1, G_2)$  of subgraphs of  $G$ , such that  $G = G_1 \cup G_2$ . In particular, there is no edge of  $G$  between  $V(G_1) - V(G_2)$  and  $V(G_2) - V(G_1)$ .

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<sup>1</sup>One can prove this bound using a naive application of the Lipton-Tarjan planar separator theorem.

**Lemma 4.** Fix  $\epsilon \in (0, 1)$  and  $c \geq 1$ . Let  $G$  be a graph with  $n$  vertices. Fix a layering  $V_0, V_1, \dots, V_p$  of  $G$ . Assume that, for every set  $B \subseteq V(G)$ , there is a separation  $(G_1, G_2)$  of  $G$  such that:

- each layer  $V_i$  contains at most  $c$  vertices in  $V(G_1) \cap V(G_2) \cap B$ , and
- both  $V(G_1) - V(G_2)$  and  $V(G_2) - V(G_1)$  contain at most  $(1 - \epsilon)|B|$  vertices in  $B$ .

Then  $\pi(G) \leq 4c(1 + \log_{1/(1-\epsilon)} n)$ .

*Proof.* Run the following recursive algorithm  $\text{COMPUTE}(V(G), 1)$ .

$\text{COMPUTE}(B, d)$

1. If  $B = \emptyset$  then exit.
2. Let  $(G_1, G_2)$  be a separation of  $G$  such that each layer  $V_i$  contains at most  $c$  vertices in  $V(G_1) \cap V(G_2) \cap B$ , and both  $V(G_1) - V(G_2)$  and  $V(G_2) - V(G_1)$  contain at most  $(1 - \epsilon)|B|$  vertices in  $B$ .
3. Let  $\text{depth}(v) := d$  for each vertex  $v \in V(G_1) \cap V(G_2) \cap B$ .
4. For  $i \in [1, p]$ , injectively label the vertices in  $V_i \cap V(G_1) \cap V(G_2) \cap B$  by  $1, 2, \dots, c$ . Let  $\text{label}(v)$  be the label assigned to each vertex  $v \in V_i \cap V(G_1) \cap V(G_2) \cap B$ .
5.  $\text{COMPUTE}((V(G_1) - V(G_2)) \cap B, d + 1)$
6.  $\text{COMPUTE}((V(G_2) - V(G_1)) \cap B, d + 1)$

The recursive application of  $\text{COMPUTE}$  determines a rooted binary tree  $T$ , where each node of  $T$  corresponds to one call to  $\text{COMPUTE}$ . Associate each vertex whose depth and label is computed in a particular call to  $\text{COMPUTE}$  with the corresponding node of  $T$ . (Observe that the depth and label of each vertex is determined exactly once.)

Colour each vertex  $v$  by  $(\text{col}(v), \text{depth}(v), \text{label}(v))$ , where  $\text{col}$  is the 4-colouring from Lemma 3. Suppose on the contrary that  $(v_1, v_2, \dots, v_{2t})$  is a repetitively coloured path in  $G$ . By Lemma 3,  $(v_1, v_2, \dots, v_t)$  and  $(v_{t+1}, v_{t+2}, \dots, v_{2t})$  have the same layer pattern. In addition,  $\text{depth}(v_i) = \text{depth}(v_{t+i})$  and  $\text{label}(v_i) = \text{label}(v_{t+i})$  for all  $i \in [1, t]$ . Let  $v_i$  and  $v_{t+i}$  be vertices in this path with minimum depth. Since  $v_i$  and  $v_{t+i}$  are in the same layer and have the same label, these two vertices were not labelled at the same step of the algorithm. Let  $x$  and  $y$  be the two nodes of  $T$  respectively associated with  $v_i$  and  $v_{t+i}$ . Let  $z$  be the least common ancestor of  $x$  and  $y$  in  $T$ . Say node  $z$  corresponds to call  $\text{COMPUTE}(B, d)$ . Thus  $v_i$  and  $v_{t+i}$  are in  $B$  (since if a vertex  $v$  is in  $B$  in the call to  $\text{COMPUTE}$  associated with some node  $q$  of  $T$ , then  $v$  is in  $B$  in the call to  $\text{COMPUTE}$  associated with each ancestor of  $q$  in  $T$ ). Let  $(G_1, G_2)$  be the separation in  $\text{COMPUTE}(B, d)$ . Since  $\text{depth}(v_i) = \text{depth}(v_{t+i}) > d$ ,

neither  $v_i$  nor  $v_{t+i}$  are in  $V(G_1) \cap V(G_2)$ . Since  $z$  is the least common ancestor of  $x$  and  $y$ , without loss of generality,  $v_i \in V(G_1) - V(G_2)$  and  $v_{t+i} \in V(G_2) - V(G_1)$ . Thus some vertex  $v_j$  in the subpath  $(v_{i+1}, v_{i+2}, \dots, v_{t+i-1})$  is in  $V(G_1) \cap V(G_2)$ . If  $v_j \in B$  then  $\text{depth}(v_j) = d$ . If  $v_j \notin B$  then  $\text{depth}(v_j) < d$ . In both cases,  $\text{depth}(v_j) < \text{depth}(v_i) = \text{depth}(v_{t+i})$ , which contradicts the choice of  $v_i$  and  $v_{t+i}$ . Hence there is no repetitively coloured path in  $G$ .

Observe that the maximum depth is at most  $1 + \log_{1/(1-\epsilon)} n$ . Therefore the number of colours is at most  $4c(1 + \log_{1/(1-\epsilon)} n)$ .  $\square$

We now prove that the condition in Lemma 4 holds for *plane triangulations*; that is, embedded planar graphs in which every face is a triangle. If  $r$  is a vertex of a connected graph  $G$  and  $V_i$  is the set of vertices in  $G$  at distance  $i$  from  $r$ , then  $V_0, V_1, V_2, \dots$  is a layering of  $G$ , called the *layering starting* at  $r$ . Observe that for each vertex  $v \in V_i$  there is a  $vr$ -path that contains exactly one vertex from each layer  $V_0, V_1, \dots, V_i$ ; we call this a *monotone* path.

**Lemma 5.** *Let  $r$  be a vertex in a plane triangulation  $G$ . Let  $V_0, V_1, \dots, V_p$  be the layering of  $G$  starting at  $r$ . For every set  $B \subseteq V(G)$ , there is a separation  $(G_1, G_2)$  of  $G$  such that:*

- *each layer  $V_i$  contains at most two vertices in  $V(G_1) \cap V(G_2) \cap B$ ,*
- *both  $V(G_1) - V(G_2)$  and  $V(G_2) - V(G_1)$  contain at most  $\frac{2}{3}|B|$  vertices in  $B$ .*

*Proof.* If  $|B| \leq 2$  then  $G_1 := G_2 := G$  satisfy the claim. Now assume that  $|B| \geq 3$ . A *lollipop*  $S$  of *height*  $k$  is a walk in  $G$  such that:

- either  $S = (u_0, u_1, \dots, u_{k-1}, u_k, v_k, v_{k-1}, \dots, v_1, v_0)$  as in Figure 1(a), or  $S = (u_0, u_1, \dots, u_{k-1}, u_k, u_{k+1}, v_k, v_{k-1}, \dots, v_1, v_0)$  as in Figure 1(b), where  $u_i, v_i \in V_i$  for each  $i \in [1, k]$ , and  $u_{k+1} \in V_{k+1}$ ;
- $u_0 = v_0 = r$  and  $u_k \neq v_k$ ; and
- if  $u_i = v_i$  for some  $i \in [1, k-1]$ , then  $u_j = v_j$  for each  $j \in [0, i]$ .

Consider a lollipop  $S$ . We define vertices to the right and left of  $S$  as follows. Let  $i \geq 0$  be the maximum index for which  $u_i = v_i$ . Let  $C_S$  be the cycle obtained from  $S$  by removing  $u_0, u_1, \dots, u_{i-1}$  ( $= v_0, v_1, \dots, v_{i-1}$ ). Then  $w$  is *to the right* of  $S$  if it is to the right of  $C_S$  when traversing  $C_S$  so that vertex  $u_{i+1}$  is visited immediately after vertex  $u_i$ . A vertex  $w$  of  $G$  is *to the left* of  $S$  if it is neither to the right of  $S$  nor a vertex of  $S$ . For the given set  $B \subseteq V(G)$ , let  $R_B(S)$  and  $L_B(S)$  be the sets of vertices in  $B$  to the right and left of  $S$ , respectively. Let  $r_B(S) := |R_B(S)|$  and  $\ell_B(S) := |L_B(S)|$ . We drop the subscript  $B$  when  $B = V(G)$ . By the Jordan Curve Theorem,  $R_B(S)$  and  $L_B(S)$  are disjoint. Note that the reverse sequence  $\overleftarrow{S}$  is also a lollipop, and  $L_B(S) = R_B(\overleftarrow{S})$  and  $R_B(S) = L_B(\overleftarrow{S})$ .

Let  $S$  be a lollipop such that:

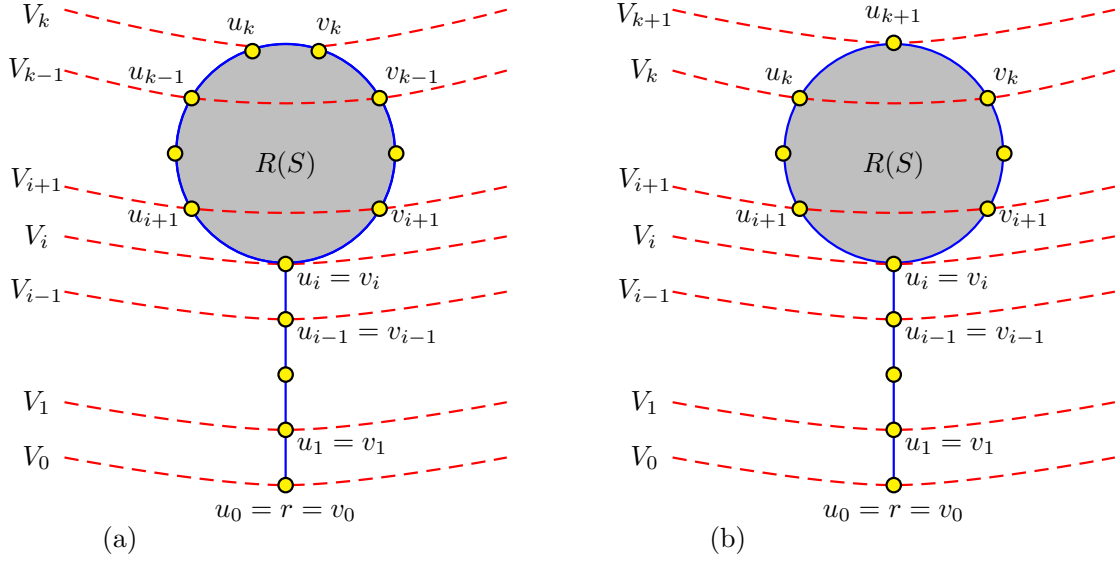


Figure 1: Two lollipops of height  $k$ . Note that the layers might have a more complicated structure than that shown here.

- (1)  $r_B(S) \leq \frac{2}{3}|B|$ ;
- (2) subject to (1),  $r_B(S)$  is maximum; and
- (3) subject to (1) and (2),  $\ell(S)$  is minimum.
- (4) subject to (1), (2) and (3),  $r(S)$  is maximum.

A lollipop satisfying (1) always exists, since if  $(r, u_1, v_1)$  is a face in clockwise order, then  $S := (u_0 = r, u_1, v_1, v_0 = r)$  is a lollipop of height 1 with  $r_B(S) = 0 \leq \frac{2}{3}|B|$ .

Say  $S$  has height  $k$ . Suppose, for the sake of contradiction, that  $\ell_B(S) > \frac{2}{3}|B|$ . Thus  $R_B(S) < \frac{1}{3}|B|$ . We distinguish the following cases:

*Case 1.*  $S = (u_0, u_1, \dots, u_{k-1}, u_k, v_k, v_{k-1}, \dots, v_1, v_0)$ : Let  $w$  be the vertex that forms a face  $f = (u_k, w, v_k)$  in clockwise order. By the definition of layering,  $w \in V_{k+1} \cup V_k \cup V_{k-1}$ .

*Case 1(a).*  $w \in V_{k+1}$ : Then  $S' := (u_0, u_1, \dots, u_{k-1}, u_k, w, v_k, v_{k-1}, \dots, v_1, v_0)$  is a lollipop. Since  $f$  is a face,  $R_B(S') = R_B(S)$  and  $L(S') = L(S) - \{w\}$ , contradicting (3).

*Case 1(b).*  $w = u_{k-1}$ : Observe that  $S' := (u_0, u_1, \dots, u_{k-1}, v_k, v_{k-1}, \dots, v_1, v_0)$  is a lollipop of height  $k-1$ . We have  $r_B(S') \leq r_B(S) + 1 < \frac{1}{3}|B| + 1 \leq \frac{2}{3}|B|$  (since  $|B| \geq 3$ ). Hence  $S'$  satisfies (1). If  $u_k \in B$  then  $r_B(S') > r_B(S)$ , contradicting (2). Now assume that  $u_k \notin B$ . We have  $\ell(S') = \ell(S)$  and  $r(S') = r(S) + 1$ , contradicting (4).

*Case 1(c).*  $w = v_{k-1}$ : This case is analogous to Case 1(b) except that we use  $S' := (u_0, u_1, \dots, u_{k-1}, u_k, v_{k-1}, \dots, v_1, v_0)$ .

*Case 1(d).*  $w \in V_{k-1} \setminus \{u_{k-1}, v_{k-1}\}$ : There is a monotone path  $P = (w = z_{k-1}, z_{k-2}, \dots, z_0 = r)$  such that if some  $z_i = u_i$  then  $z_j = u_j$  for each  $j \in [0, i]$ , and if some  $z_i = v_i$  then  $z_j = v_j$  for each  $j \in [0, i]$ . Observe that both  $S' := (u_0, u_1, \dots, u_{k-1}, u_k, z_{k-1}, z_{k-2}, \dots, z_1, z_0)$  and  $S'' := (z_0, z_1, \dots, z_{k-1}, v_k, v_{k-1}, \dots, v_1, v_0)$  are lollipops. By assumption,  $r_B(\overleftarrow{S}) = \ell_B(S) > \frac{2}{3}|B|$ . We have  $r_B(S') > \frac{2}{3}|B|$  since  $S'$  is a lollipop with  $r_B(S') \geq r_B(S)$  and  $\ell(S') < \ell(S)$ . Similarly,  $r_B(S'') > \frac{2}{3}|B|$ . Hence  $r_B(\overleftarrow{S}) + r_B(S') + r_B(S'') > 2|B|$ . Thus  $R_B(\overleftarrow{S}) \cap R_B(S') \cap R_B(S'') \neq \emptyset$ , which is a contradiction since  $R_B(S') \cap R_B(S'') \subseteq R_B(S) = L_B(\overleftarrow{S})$ .

*Case 1(e).*  $w \in V_k$ : This case is analogous to Case 1(d), except that here  $P$  is a monotone path  $(w = z_k, z_{k-1}, \dots, z_0 = r)$ , and  $S' := (u_0, u_1, \dots, u_{k-1}, u_k, z_k, z_{k-1}, \dots, z_1, z_0)$  and  $S'' := (z_0, z_1, \dots, z_{k-1}, z_k, v_k, v_{k-1}, \dots, v_1, v_0)$ .

*Case 2.*  $S = (u_0, u_1, \dots, u_{k-1}, u_k, u_{k+1}, v_k, v_{k-1}, \dots, v_1, v_0)$ : Let  $w$  be the vertex that forms a face  $f = (u_k, w, u_{k+1})$  in clockwise order. Hence  $w \in V_{k+1} \cup V_k$ .

*Case 2(a).*  $w \in V_{k+1}$ : This case is analogous to Case 1(a) except that  $S' := (u_0, u_1, \dots, u_{k-1}, u_k, w, u_{k+1}, v_k, v_{k-1}, \dots, v_1, v_0)$ .

*Case 2(b).*  $w = v_k$ : This case is analogous to Case 1(c) except with  $S' := (u_0, u_1, \dots, u_{k-1}, u_k, v_k, v_{k-1}, \dots, v_1, v_0)$ .

*Case 2(c).*  $w \in V_k - \{v_k\}$ : This case is analogous to Case 1(d), except that here  $P$  is a monotone path  $(w = z_k, z_{k-1}, \dots, z_0)$ , and  $S' := (u_0, u_1, \dots, u_{k-1}, u_k, z_k, z_{k-1}, \dots, z_1, z_0)$  and  $S'' := (z_0, z_1, \dots, z_{k-1}, z_k, u_{k+1}, v_k, v_{k-1}, \dots, v_1, v_0)$ .

Each case leads to a contradiction. Hence  $r_B(S) \leq \frac{2}{3}|B|$  and  $\ell_B(S) \leq \frac{2}{3}|B|$ . Let  $G_1$  be the subgraph induced by the vertices in  $S$  and to the right of  $S$ . Let  $G_2$  be the subgraph induced by the vertices in  $S$  and to the left of  $S$ . By the Jordan Curve Theorem, no vertex to the right of  $S$  is adjacent to a vertex to the left of  $S$ . Hence  $G = G_1 \cup G_2$  and  $(G_1, G_2)$  is the desired separation.  $\square$

Lemmas 4 and 5 together prove Theorem 1 (since every planar graph with at least four vertices is a spanning subgraph of a plane triangulation).

### 3 Proof of Theorem 2

Theorem 2 is a special case of the following result with  $H = K_5$  or  $H = K_{3,3}$ . A graph  $H$  is *apex* if  $H - v$  is planar for some vertex  $v$  of  $H$ .

**Theorem 6.** *For every fixed apex graph  $H$  there is a constant  $c = c(H)$  such that, for every*

integer  $k$ , every  $H$ -minor-free graph  $G$  is  $c^{k^2}$ -colourable such that  $G$  contains no repetitively coloured path of order at most  $2k$ .

*Proof.* Eppstein [17] proved that for some function  $f$  (depending on  $H$ ), for every  $H$ -minor-free graph  $G$ , for every vertex  $r$  of  $G$ , and for every integer  $\ell \geq 0$ , the set of vertices in  $G$  at distance at most  $\ell$  from  $r$  induces a subgraph of treewidth at most  $f(\ell)$ . This is called the *diameter-treewidth* or *bounded local treewidth* property; also see [14, 15, 20]. Demaine and Hajiaghayi [16] strengthened Eppstein's result by showing that one can take  $f(\ell) = c\ell$  for some constant  $c = c(H)$ .

Let  $G$  be an  $H$ -minor-free graph. By considering each connected component in turn, we may assume that  $G$  is connected. Let  $r$  be a vertex of  $G$ . Let  $V_0, V_1, \dots, V_p$  be the layering of  $G$  starting at some vertex  $r$  of  $G$ . Fix an integer  $k \geq 1$ . For  $i \in [1, p]$ , let  $G_i := G[V_i \cup V_{i+1} \cup \dots \cup V_{\min\{p, i+2k-1\}}]$ , and let  $G'_i$  be the minor of  $G$  obtained by contracting the connected subgraph  $G[V_0 \cup V_1 \cup \dots \cup V_{i-1}]$  into a single vertex  $r_i$ . Thus  $G'_i$  is an  $H$ -minor-free graph containing  $G_i$  as a subgraph, and each vertex in  $G_i$  is at distance at most  $2k$  from  $r_i$  in  $G'_i$ . By the diameter-treewidth property,  $G_i$  has treewidth at most  $2ck$ . By a theorem of Kündgen and Pelsmayer [30], there is a nonrepetitive  $4^{2ck}$ -colouring  $\psi_i$  of  $G_i$ .

For each vertex  $v$  of  $G$ , define  $\psi(v) := (\phi_0(v), \phi_1(v), \dots, \phi_{2k-1}(v))$ , where  $\phi_j(v) := \psi_i(v)$  and  $i$  is the unique integer for which  $i \equiv j \pmod{2k}$  and  $v \in V(G_i)$ . Suppose on the contrary that  $G$  contains a repetitively coloured path  $P = (v_1, \dots, v_{2t})$  of order at most  $2k$  (under the colouring  $\psi$ ). Thus  $P$  is contained in some  $G_i$ . Let  $j := i \bmod 2k$ . Hence  $\psi_i(v_a) = \phi_j(v_a) = \phi_j(v_{t+a}) = \psi_i(v_{t+a})$  for each  $a \in [1, t]$ . That is,  $P$  is repetitively coloured by  $\psi_i$  in the colouring of  $G_i$ . This contradiction proves that  $G$  contains no repetitively coloured path under  $\psi$ . The number of colours is  $(4^{2ck})^{2k} = (4^{4c})^{k^2}$ .  $\square$

Graphs embeddable on a fixed surface exclude a fixed apex graph as a minor [17]. Thus Theorem 6 implies:

**Corollary 7.** *For every fixed surface  $\Sigma$  there is a constant  $c = c(\Sigma)$  such that, for every integer  $k \geq 1$ , every graph  $G$  embeddable in  $\Sigma$  is  $c^{k^2}$ -colourable such that  $G$  contains no repetitively coloured path of order at most  $2k$ .*

## 4 Open Problems

Our research suggests two open problems:

1. Is  $\pi(G) \in o(\log n)$  for every planar graph  $G$  with  $n$  vertices?
2. Is there a polynomial function  $f$  such that for every integer  $k \geq 1$  every planar graph  $G$  is  $f(k)$ -colourable such that  $G$  contains no repetitively coloured path of order at most  $2k$ ?

Finally, we mention a class of planar graphs that seem difficult to nonrepetitively colour. Let  $T$  be a tree rooted at a vertex  $r$ . Let  $V_i$  be the set of vertices in  $T$  at distance  $i$  from  $r$ . Draw  $T$  in the plane with no crossings. Add a cycle on each  $V_i$  in the cyclic order defined by the drawing to create a planar graph  $G_T$ . It is open whether  $\pi(G_T) \leq c$  for some constant  $c$  independent of  $T$ . Note that this class of planar graphs includes examples with unbounded degree and unbounded treewidth.

## References

- [1] MICHAEL O. ALBERTSON, GLENN G. CHAPPELL, HAL A. KIERSTEAD, ANDRÉ KÜNDGEN, AND RADHIKA RAMAMURTHI. Coloring with no 2-colored  $P_4$ 's. *Electron. J. Combin.*, 11 #R26, 2004. [http://www.combinatorics.org/Volume\\_11/Abstracts/v11i1r26.html](http://www.combinatorics.org/Volume_11/Abstracts/v11i1r26.html). MR: 2056078.
- [2] NOGA ALON AND JAROSŁAW GRYTCZUK. Breaking the rhythm on graphs. *Discrete Math.*, 308:1375–1380, 2008. doi: [10.1016/j.disc.2007.07.063](https://doi.org/10.1016/j.disc.2007.07.063). MR: 2392054.
- [3] NOGA ALON, JAROSŁAW GRYTCZUK, MARIUSZ HAŁUSZCZAK, AND OLIVER RIORDAN. Nonrepetitive colorings of graphs. *Random Structures Algorithms*, 21(3-4):336–346, 2002. doi: [10.1002/rsa.10057](https://doi.org/10.1002/rsa.10057). MR: 1945373.
- [4] JÁNOS BARÁT AND JÚLIUS CZAP. Vertex coloring of plane graphs with nonrepetitive boundary paths. 2011. arXiv: [1105.1023](https://arxiv.org/abs/1105.1023).
- [5] JÁNOS BARÁT AND PÉTER P. VARJÚ. On square-free vertex colorings of graphs. *Studia Sci. Math. Hungar.*, 44(3):411–422, 2007. doi: [10.1556/SScMath.2007.1029](https://doi.org/10.1556/SScMath.2007.1029). MR: 2361685.
- [6] JÁNOS BARÁT AND PÉTER P. VARJÚ. On square-free edge colorings of graphs. *Ars Combin.*, 87:377–383, 2008. MR: 2414029.
- [7] JÁNOS BARÁT AND DAVID R. WOOD. Notes on nonrepetitive graph colouring. *Electron. J. Combin.*, 15:R99, 2008. [http://www.combinatorics.org/Volume\\_15/Abstracts/v15i1r99.html](http://www.combinatorics.org/Volume_15/Abstracts/v15i1r99.html). MR: 2426162.
- [8] BOŠTJAN BREŠAR, JAROSŁAW GRYTCZUK, SANDI KLAVŽAR, STANISŁAW NIWCZYK, AND IZTOK PETERIN. Nonrepetitive colorings of trees. *Discrete Math.*, 307(2):163–172, 2007. doi: [10.1016/j.disc.2006.06.017](https://doi.org/10.1016/j.disc.2006.06.017). MR: 2285186.
- [9] BOŠTJAN BREŠAR AND SANDI KLAVŽAR. Square-free colorings of graphs. *Ars Combin.*, 70:3–13, 2004. MR: 2023057.
- [10] PANAGIOTIS CHEILARIS, ERNST SPECKER, AND STATHIS ZACHOS. Neochromatica. *Comment. Math. Univ. Carolin.*, 51(3):469–480, 2010. <http://www.dml.cz/dmlcz/140723>. MR: 2741880.



- [11] JAMES D. CURRIE. There are ternary circular square-free words of length  $n$  for  $n \geq 18$ . *Electron. J. Combin.*, 9(1), 2002. [http://www.combinatorics.org/Volume\\_9/Abstracts/v9i1n10.html](http://www.combinatorics.org/Volume_9/Abstracts/v9i1n10.html). MR: 1936865.
- [12] JAMES D. CURRIE. Pattern avoidance: themes and variations. *Theoret. Comput. Sci.*, 339(1):7–18, 2005. doi: [10.1016/j.tcs.2005.01.004](https://doi.org/10.1016/j.tcs.2005.01.004). MR: 2142070.
- [13] SEBASTIAN CZERWIŃSKI AND JAROSŁAW GRYTCZUK. Nonrepetitive colorings of graphs. *Electron. Notes Discrete Math.*, 28:453–459, 2007. doi: [10.1016/j.endm.2007.01.063](https://doi.org/10.1016/j.endm.2007.01.063). MR: 2324051.
- [14] ERIK D. DEMAINE, FEDOR V. FOMIN, MOHAMMADTAGHI HAJIAGHAYI, AND DIMITRIOS M. THILIKOS. Bidimensional parameters and local treewidth. *SIAM J. Discrete Math.*, 18(3):501–511, 2004/05. doi: [10.1137/S0895480103433410](https://doi.org/10.1137/S0895480103433410). MR: 2134412.
- [15] ERIK D. DEMAINE AND MOHAMMADTAGHI HAJIAGHAYI. Diameter and treewidth in minor-closed graph families, revisited. *Algorithmica*, 40(3):211–215, 2004. doi: [10.1007/s00453-004-1106-1](https://doi.org/10.1007/s00453-004-1106-1). MR: 2080518.
- [16] ERIK D. DEMAINE AND MOHAMMADTAGHI HAJIAGHAYI. Equivalence of local treewidth and linear local treewidth and its algorithmic applications. In *Proc. 15th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '04)*, pp. 840–849. SIAM, 2004. <http://dl.acm.org/citation.cfm?id=982792.982919>.
- [17] DAVID EPPSTEIN. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3-4):275–291, 2000. doi: [10.1007/s004530010020](https://doi.org/10.1007/s004530010020). MR: 1759751.
- [18] GUILLAUME FERTIN, ANDRÉ RASPAUD, AND BRUCE REED. Star coloring of graphs. *J. Graph Theory*, 47(3):163–182, 2004. doi: [10.1002/jgt.20029](https://doi.org/10.1002/jgt.20029). MR: 2089462.
- [19] FRANCESCA FIORENZI, PASCAL OCHEM, PATRICE OSSONA DE MENDEZ, AND XUDING ZHU. Thue choosability of trees. *Discrete Applied Math.*, 159(17):2045–2049, 2011. doi: [10.1016/j.dam.2011.07.017](https://doi.org/10.1016/j.dam.2011.07.017). MR: 2832329.
- [20] MARTIN GROHE. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003. doi: [10.1007/s00493-003-0037-9](https://doi.org/10.1007/s00493-003-0037-9). MR: 2046826.
- [21] JAROSŁAW GRYTCZUK. Thue-like sequences and rainbow arithmetic progressions. *Electron. J. Combin.*, 9(1):R44, 2002. [http://www.combinatorics.org/Volume\\_9/Abstracts/v9i1r44.html](http://www.combinatorics.org/Volume_9/Abstracts/v9i1r44.html). MR: 1946146.
- [22] JAROSŁAW GRYTCZUK. Nonrepetitive colorings of graphs—a survey. *Int. J. Math. Math. Sci.*, 74639, 2007. doi: [10.1155/2007/74639](https://doi.org/10.1155/2007/74639). MR: 2272338.
- [23] JAROSŁAW GRYTCZUK. Nonrepetitive graph coloring. In *Graph Theory in Paris*, Trends in Mathematics, pp. 209–218. Birkhauser, 2007.

- [24] JAROSŁAW GRYTCZUK. Thue type problems for graphs, points, and numbers. *Discrete Math.*, 308(19):4419–4429, 2008. doi: [10.1016/j.disc.2007.08.039](https://doi.org/10.1016/j.disc.2007.08.039). MR: [2433769](#).
- [25] JAROSŁAW GRYTCZUK, JAKUB KOZIK, AND PIOTR MICEK. A new approach to non-repetitive sequences. 2011. arXiv: [1103.3809](https://arxiv.org/abs/1103.3809). To appear in *Random Structures Algorithms*.
- [26] JAROSŁAW GRYTCZUK, JAKUB PRZYBYŁO, AND XUDING ZHU. Nonrepetitive list colourings of paths. *Random Structures Algorithms*, 38(1-2):162–173, 2011. doi: [10.1002/rsa.20347](https://doi.org/10.1002/rsa.20347). MR: [2768888](#).
- [27] JOCHEN HARANTA AND STANISLAV JENDROL'. Nonrepetitive vertex colorings of graphs. *Discrete Math.*, 312(2):374–380, 2012. doi: [10.1016/j.disc.2011.09.027](https://doi.org/10.1016/j.disc.2011.09.027).
- [28] FRÉDÉRIC HAVET, STANISLAV JENDROŤ, ROMAN SOTÁK, AND ERIKA ŠKRABUĽÁKOVÁ. Facial non-repetitive edge-coloring of plane graphs. *J. Graph Theory*, 66(1):38–48, 2011. doi: [10.1002/jgt.20488](https://doi.org/10.1002/jgt.20488). MR: [2742187](#).
- [29] STANISLAV JENDROL AND ERIKA ŠKRABUĽÁKOVÁ. Facial non-repetitive edge colouring of semiregular polyhedra. *Acta Univ. M. Belii Ser. Math.*, 15:37–52, 2009. <http://actamath.savbb.sk/acta1503.shtml>. MR: [2589669](#).
- [30] ANDRE KÜNDGEN AND MICHAEL J. PELSMAYER. Nonrepetitive colorings of graphs of bounded tree-width. *Discrete Math.*, 308(19):4473–4478, 2008. doi: [10.1016/j.disc.2007.08.043](https://doi.org/10.1016/j.disc.2007.08.043). MR: [2433774](#).
- [31] FEDOR MANIN. The complexity of nonrepetitive edge coloring of graphs, 2007. arXiv: [0709.4497](https://arxiv.org/abs/0709.4497).
- [32] DÁNIEL MARX AND MARCUS SCHAEFER. The complexity of nonrepetitive coloring. *Discrete Appl. Math.*, 157(1):13–18, 2009. doi: [10.1016/j.dam.2008.04.015](https://doi.org/10.1016/j.dam.2008.04.015). MR: [2479374](#).
- [33] JAROSLAV NEŠETŘIL AND PATRICE OSSONA DE MENDEZ. Colorings and homomorphisms of minor closed classes. In BORIS ARONOV, SAUGATA BASU, JÁNOS PACH, AND MICHA SHARIR, eds., *Discrete and Computational Geometry, The Goodman-Pollack Festschrift*, vol. 25 of *Algorithms and Combinatorics*, pp. 651–664. Springer, 2003. MR: [2038495](#).
- [34] JAROSLAV NEŠETŘIL, PATRICE OSSONA DE MENDEZ, AND DAVID R. WOOD. Characterisations and examples of graph classes with bounded expansion. *European J. Combinatorics*, 33(3):350–373, 2011. doi: [10.1016/j.ejc.2011.09.008](https://doi.org/10.1016/j.ejc.2011.09.008).
- [35] WESLEY PEGDEN. Highly nonrepetitive sequences: winning strategies from the local lemma. *Random Structures Algorithms*, 38(1-2):140–161, 2011. doi: [10.1002/rsa.20354](https://doi.org/10.1002/rsa.20354). MR: [2768887](#)

- [36] ANDRZEJ PEZARSKI AND MICHAŁ ZMARZ. Non-repetitive 3-coloring of subdivided graphs. *Electron. J. Combin.*, 16(1):N15, 2009. [http://www.combinatorics.org/Volume\\_16/Abstracts/v16i1n15.html](http://www.combinatorics.org/Volume_16/Abstracts/v16i1n15.html). MR: 2515755.
- [37] NARAD RAMPERSAD. A note on non-repetitive colourings of planar graphs. 2003. arXiv: [math/0307365](https://arxiv.org/abs/math/0307365).
- [38] AXEL THUE. Über unendliche Zeichenreihen. *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiania*, 7:1–22, 1906.
- [39] DAVID R. WOOD. Acyclic, star and oriented colourings of graph subdivisions. *Discrete Math. Theor. Comput. Sci.*, 7(1):37–50, 2005. <http://www.dmtcs.org/dmtcs-ojs/index.php/dmtcs/article/view/60>. MR: 2164057.

## A Lower Bounds

Barát and Varjú [5] constructed a planar graph  $G$  with  $\pi(G) \geq 10$ . Pascal Ochem [private communication] observed that this lower bound can be improved to 11 by adapting a construction due to Albertson et al. [1] as follows. Barát and Varjú [5] constructed an outerplanar graph  $H$  with  $\pi(H) \geq 7$ . Let  $G$  be the following planar graph. Start with a path  $P = (v_1, \dots, v_{22})$ . Add two adjacent vertices  $x$  and  $y$  that both dominate  $P$ . Let each vertex  $v_i$  in  $P$  be adjacent to every vertex in a copy  $H_i$  of  $H$ . Suppose on the contrary that  $G$  is nonrepetitively 10-colourable. Without loss of generality,  $x$  and  $y$  are respectively coloured 1 and 2. A vertex in  $P$  is *redundant* if its colour is used on some other vertex in  $P$ . If no two adjacent vertices in  $P$  are redundant then at least 11 colours appear exactly once on  $P$ , which is a contradiction. Thus some pair of consecutive vertices  $v_i$  and  $v_{i+1}$  in  $P$  are redundant. Without loss of generality,  $v_i$  and  $v_{i+1}$  are respectively coloured 3 and 4. If some vertex in  $H_i \cup H_{i+1}$  is coloured 1 or 2, then since  $v_i$  and  $v_{i+1}$  are redundant, with  $x$  or  $y$  we have a repetitively coloured path on 4 vertices. Now assume that no vertex in  $H_i \cup H_{i+1}$  is coloured 1 or 2. If some vertex in  $H_i$  is coloured 4 and some vertex in  $H_{i+1}$  is coloured 3, then with  $v_i$  and  $v_{i+1}$ , we have a repetitively coloured path on 4 vertices. Thus no vertex in  $H_i$  is coloured 4 or no vertex in  $H_{i+1}$  is coloured 3. Without loss of generality, no vertex in  $H_i$  is coloured 4. Since  $v_i$  dominates  $H_i$ , no vertex in  $H_i$  is coloured 3. We have proved that no vertex in  $H_i$  is coloured 1, 2, 3 or 4, which is a contradiction, since  $\pi(H_i) \geq 7$ . Therefore  $\pi(G) \geq 11$ .